ON THE RAINBOW CONNECTION OF LINE, MIDDLE, AND TOTAL OF WHEEL

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Abstract. An edge-colored graph G is called rainbow connected if any two vertices in G are connected by a path whose no two its edges are colored the same. The rainbow connection of G, denoted by $rc(G)$, is the smallest number of colors that are needed such as G be a rainbow connected graph. An edge-colored graph G is called strong rainbow connected if any two vertices in G are connected by a geodesic whose no two its edges are colored the same. The strong rainbow connection for G, denoted by $src(G)$, is the smallest number of colors that are needed such as G be a strong rainbow connected graph.

In this paper, we determine rainbow connection number and strong number connectin of line, middle and total of wheel.

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1. Introduction

The concept of rainbow connection was introduced by Gary Chartrand et al. 2008. For a nontrivial connected graph G and a positive integer k, let $c : E(G) \to \{1, 2, ..., k\}$ be an edge coloring of G, where the adjacent edges can be colored the same. A path in G is called a rainbow path if no two its edges are colored the same. G is called a rainbow-connected if every two vertices x and y in G, there exists a rainbow $x - y$ path. In this case, the coloring c is a rainbow coloring. If c is a rainbow coloring with k colors are used, then c is a rainbow k-coloring. If k is the smallest number, then k is rainbow connection number $rc(G)$ of G. Clearly $diam(G) \leq rc(G)$, where $diam(G)$ is the diameter of G.

Let c an edge coloring of a nontrivial graph G. For two vertices x and y of G, a rainbow $x-y$ geodesic in G is a $x-y$ rainbow path of length $d(x, y)$. The graph G is called a strongly rainbow-connected if every two vertices x and y in G, there exists a rainbow $x - y$ geodesic. In this case, the coloring c is called a strong rainbow coloring of G. The smallest positive integer k for which G has a strong rainbow k-coloring is the strong rainbow connection number of G, denoted by $src(G)$.

Gary Chartrand et al. [2] provide that if G is a nontrivial connected graph with size m, then

$$diam(G) \leq rc(G) \leq src(G) \leq m.$$

In [2] Gary Chartrand et al. determined the rainbow connection number of some classes of graphs.

2. Preliminary Notes

Definition 2.1. *The line graph of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.*

Definition 2.2. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of a graph G, denoted by $M(G)$, is defined as follows.The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in$M(G)$ in case one of the following holds:*

(1) x, y are in $E(G)$ and x, y are adjacent in G.
(2) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Definition 2.3. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of a graph G, denoted by $T(G)$, is defined as follows.The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in$T(G)$ in case one of the following holds:*

(1) x, y are in $V(G)$ and x is adjacent to y in G.
(2) x, y are in $E(G)$ and x, y are adjacent in G.
(3) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

3. Main Results

The definition of Line Wheel as follows,

$$V(L(W_n)) = \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\} \tag{3.1}$$

$$\begin{aligned} E(L(W_n)) = \ & \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_i w_i \mid 1 \leq i \leq n\} \\ & \{w_i v_{i+1} \mid 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{w_i w_{i+1} \mid 1 \leq i \leq n, w_{n+1} = w_1\}. \end{aligned} \tag{3.2}$$

The definition of Middle Wheel as follows,

$$V(M(W_n)) = \{u_i \mid 0 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\} \tag{3.3}$$

$$\begin{aligned} E(M(W_n)) = \ & \{u_0 v_i \mid 1 \leq i \leq n\} \cup \{u_i v_i \mid 1 \leq i \leq n\} \\ & \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \{w_{i-1} u_i \mid 1 \leq i \leq n, w_0 = w_n\} \\ & \cup \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_i w_i \mid 1 \leq i \leq n\} \\ & \cup \{w_{i-1} v_i \mid 1 \leq i \leq n, w_0 = w_n\} \cup \{w_{i-1} w_i \mid 1 \leq i \leq n, w_0 = w_n\}. \end{aligned} \tag{3.4}$$

And the definition of Total Wheel as follows

$$V(T(W_n)) = \{u_i \mid 0 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\} \quad (3.5)$$

$$\begin{aligned} E(T(W_n)) = & \{u_0u_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq n, u_{n+1} = u_1\} \\ & \cup \{u_0v_i \mid 1 \leq i \leq n\} \cup \{u_iv_i \mid 1 \leq i \leq n\} \\ & \cup \{u_iw_i \mid 1 \leq i \leq n\} \cup \{w_iu_{i+1} \mid 1 \leq i \leq n, u_{n+1} = u_1\} \\ & \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_iw_i \mid 1 \leq i \leq n\} \\ & \cup \{w_iv_{i+1} \mid 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{w_iw_{i+1} \mid 1 \leq i \leq n, w_{n+1} = w_1\}. \end{aligned} \quad (3.6)$$

The following are the diameter of line, middle, and total of wheel for $n \geq 3$.

$$diam(L(W_n)) = \begin{cases} 2, & n = 3, 4 \\ 3, & n \geq 5 \end{cases}$$

$$diam(M(W_n)) = \begin{cases} 2, & n = 3 \\ 3, & n \geq 4 \end{cases}$$

$$diam(T(W_n)) = \begin{cases} 2, & n = 3, 4 \\ 3, & n \geq 5. \end{cases}$$

3.1. *Rainbow Connection of Line Wheel*

Theorem 3.1. *If $n \geq 3$ and $G = L(W_n)$ is line of wheel, then*

$$rc(G) = src(G) = \begin{cases} 2, & n = 3, 4 \\ 3, & n \geq 5 \end{cases}$$

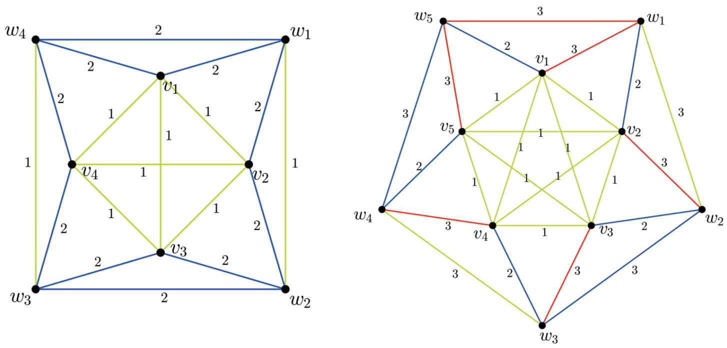

Figure 1. Rainbow coloring of line wheel W_4 line wheel W_5

Proof.

(1) Suppose that $n = 3, 4$. Since $diam(G) = 2$, then $rc(G) \geq 2$. Next, it will be shown that $rc(G) \leq 2$. Since $c_{11} : E(G) \to \{1, 2\}$ defined by

$$c_{11}(e) = \begin{cases} 1, & e \in \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{w_i w_{i+1} \mid 1 \leq i \leq n-1, i \text{ is odd}\} \\ & \quad \cup\{w_n w_1, n \text{ is odd}\} \\ 2, & e \text{ others} \end{cases}$$

is a rainbow strong coloring, it follows that $rc(G) = src(G) = 2$ for $n = 3, 4$.

(2) Suppose that $n = 5$. Since $diam(G) = 2$ for $n = 5$, then $rc(G) \geq 2$ for $n = 5$. Assume, to the contrary that $rc(G) \leq 2$, for $n = 5$. Let c_{12} is a rainbow 2-coloring. Without loss generality, assume that $c_{12}(w_1 w_2) = 1$. For $1 \leq i \leq 5$, there exists w_i, w_{i+1}, w_{i+2} with $w_{n+1} = w_1$ and $w_{n+2} = w_2$ in G which is $w_i - w_{i+2}$ path with length 2 and so, $c_{12}(w_2 w_3) = 2$. Since $c_{12}(w_2 w_3) = 2$, it follows that $c_{12}(w_3 w_4) = 1$. So $c_{12}(w_4 w_5) = 2$ and $c_{12}(w_5 w_1) = 1$. Since $c_{12}(w_5 w_1) = 1$ and $c_{12}(w_1 w_2) = 1$, there is no rainbow $w_5 - w_2$ path which is a contradiction. Therefore, $rc(G) \geq 3$.

Next, it will shown that $rc(G) \leq 3$ for $n = 5$. Let $c_{13} : E(G) \to \{1, 2, 3\}$ is an edge-coloring which is defined as follows

$$c_{13}(e) = \begin{cases} 1, & e \in \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{w_n w_1\} \\ 2, & e \in \{w_i v_{i+1} \mid 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{w_i w_{i+1} \mid 1 \leq i \leq n, i \text{ is odd}\} \\ 3, & e \text{ others} \end{cases}$$

It's clear that for each two adjacent vertices in G has a rainbow path if each edge of G is colored by c_{13}. For $1 \leq i, j \leq n$ and $a, b \in V(G)$, there exists a rainbow 3-coloring c_{13} such as there exists a rainbow $a - b$ path with $d(a, b) \geq 2$ which are considered as follow.

(a) w_i, w_{i+1}, w_j if $a = w_i$, $b = w_j$, $i < j$ and $d(a, b) = 2$.
(b) w_i, v_i, v_j, w_j if $a = w_i$, $b = w_j$ and $d(a, b) > 2$.
(c) v_i, v_j, w_j if $a = v_i$ and $b = w_j$.

Since there exists a rainbow geodesic $a - b$ path for $a, b \in V(G)$, then c_{13} is a strong rainbow 3-coloring. Therefore, $rc(G) = src(G) = 3$ for $n = 5$.

(3) Finaly, Suppose that $n > 5$. Since $diam(G) = 3$ for $n > 5$, then $rc(G) \geq 3$ for $n > 5$. Next, it will shown that $rc(G) \leq 3$ for $n > 5$. For $a, b \in V(G)$, there exists a strong rainbow 3-coloring c_{13} such as there exists a rainbow geodesic $a - b$ path Therefore, $rc(G) = src(G) = 3$ for $n > 5$. □

3.2. *Rainbow Connection of Middle Wheel*

Theorem 3.2. *If $n \geq 3$ and $G = M(W_n)$ is middle of wheel, then*

$$rc(G) = \begin{cases} 2, & n = 3 \\ 3, & 4 \leq n \leq 9 \\ 4, & n \geq 10 \end{cases}$$

Proof.

(1) Suppose that $n = 3$. Since $diam(G) = 2$, then $rc(G) \geq 2$. Next, it will shown that $rc(G) \leq 2$. There exists a rainbow 2-coloring $c_{21} : E(G) \rightarrow \{1, 2\}$ which is defined as

$$c_{21}(e) = \begin{cases} 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{u_iw_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ 2, & e \text{ others.} \end{cases} \quad (3.7)$$

Therefore, $rc(G) = 2$ for $n = 3$.

(2) Suppose that $4 \leq n \leq 9$. Since $diam(G) = 3$ for $n \geq 4$, then $rc(G) \geq 3$. Let $c_{22} : E(G) \rightarrow \{1, 2, 3\}$ is an edge coloring which is defined as follows

$$c_{22}(e) = \begin{cases} 1, & e \in \{u_0v_i \mid 4 \leq i \leq n\} \cup \{u_iv_i \mid 1 \leq i \leq 3\} \cup \{u_iw_i \mid 1 \leq i \leq n\} \cup \\ & \{v_iv_j \mid 4 \leq i, j \leq n, i \neq j)\} \cup \{v_iw_i \mid 1 \leq i \leq 3\} \cup \\ & \{v_iw_{i-1} \mid 1 \leq i \leq 3, w_0 = w_n\} \\ 2, & e \in \{u_iv_i \mid 4 \leq i \leq 6\} \cup \{w_iw_{i+1} \mid 1 \leq i \leq n, w_{n+1} = w_1\} \cup \\ & \{v_iw_i \mid 4 \leq i \leq 6\} \cup \{v_iw_{i-1} \mid 4 \leq i \leq 6\} \cup \\ & \{v_iv_j \mid 1 \leq i \leq 3, 7 \leq j \leq n\} \\ 3, & e \text{ others.} \end{cases}$$

For $a, b \in V(G)$ with $d(a, b) \geq 2$, there exists a rainbow 3-coloring c_{22} such as there exists a rainbow $a - b$ path are considered as follows:

(a) u_i, v_i, v_j, u_j or u_i, w_i, w_{j-1}, w_j where $w_0 = w_n$ if $a = u_i$ and $b = u_j$,
(b) u_i, v_i, v_j if $a = u_i$ and $b = v_j$,
(c) u_i, w_i, w_j or $u_i, v_i, v_j + 1, w_j$ where $v_{n+1} = v_1$ if $a = u_i$ and $b = w_j$,
(d) w_i, w_{j-1}, u_j, w_j or w_i, u_i, u_j, w_j if $a = w_i$ and $b = w_j$.

Therefore, $rc(G) = 3$ for $4 \leq n \leq 9$.

(3) Suppose that $n \geq 10$. Let H is a subgraph of G, where $V(H) = \{u_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\} \subset V(G)$ and $E(H) = \{u_iw_i \mid 1 \leq i \leq n\} \cup \{w_{i-1}u_i \mid 1 \leq i \leq n, w_0 = w_n\} \cup \{w_{i-1}w_i \mid 1 \leq i \leq n, w_0 = w_n\} \subset E(G)$. Let $V' = \{v_i \mid 1 \leq i \leq n\}$ and $E' = \{u_iv_i \mid 1 \leq i \leq n\}$. Asume, to the contrary that $rc(G) \geq 3$. Let c_{23} is a rainbow 3-coloring in G. So, there exist $x, y \in \{u_i \mid 1 \leq i \leq n\} \subset V(H)$ and $x', y' \in V'$ such as $d(x, y) > 3$ in H, $xx', yy' \in E'$, and xx', yy' are assigned the same. Since x, x', y', y is the only $x - y$ path which has $d(x, y) = 3$ in G, it follows that there is no rainbow $x - y$ path in G, which is a contradiction. Thus $rc(G) \geq 4$.

Next, it will shown that $rc(G) \leq 4$. Let $c_{24} : E(G) \rightarrow \{1, 2, 3, 4\}$ is an edge coloring which is defined as

$$c_{24}(e) = \begin{cases} 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ 2, & e \in \{u_iv_i \mid 1 \leq i \leq n \text{ and } i \text{ is odd}\} \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, w_0 = w_n\} \\ 3, & e \in \{u_iv_i \mid 2 \leq i \leq n \text{ and } i \text{ is even}\} \cup \{v_iw_i \mid 1 \leq i \leq n\} \\ 4, & e \text{ others.} \end{cases}$$

For $a, b \in V(G)$ with $d(a, b) \geq 2$, there exists a rainbow 4-coloring c_{24} such as the rainbow $a - b$ path are considered as follow.

(a) If $a = u_i$ and $b = u_j$

i. $a = u_i, v_i, v_j, w_j, u_j = b$ if i and j are both odd.

ii. $a = u_i, v_i, v_j, w_{j-1}, v_j = b$ if i and j are both even.

iii. $a = u_i, v_i, v_j, u_j = b$ if i is odd and j is even or i is even and j is odd.

(b) u_i, v_i, v_j, w_j or $a = u_i, v_i, v_{j+1}, w_j$ where $v_{n+1} = v_1$ if $a = v_i$ and $b = w_j$.

(c) $a = w_i, v_{i+1}, v_j, w_j = b$ if $a = w_i$ and $b = w_j$.

Therefore, $rc(G) = 4$ for $n \geq 10$. □

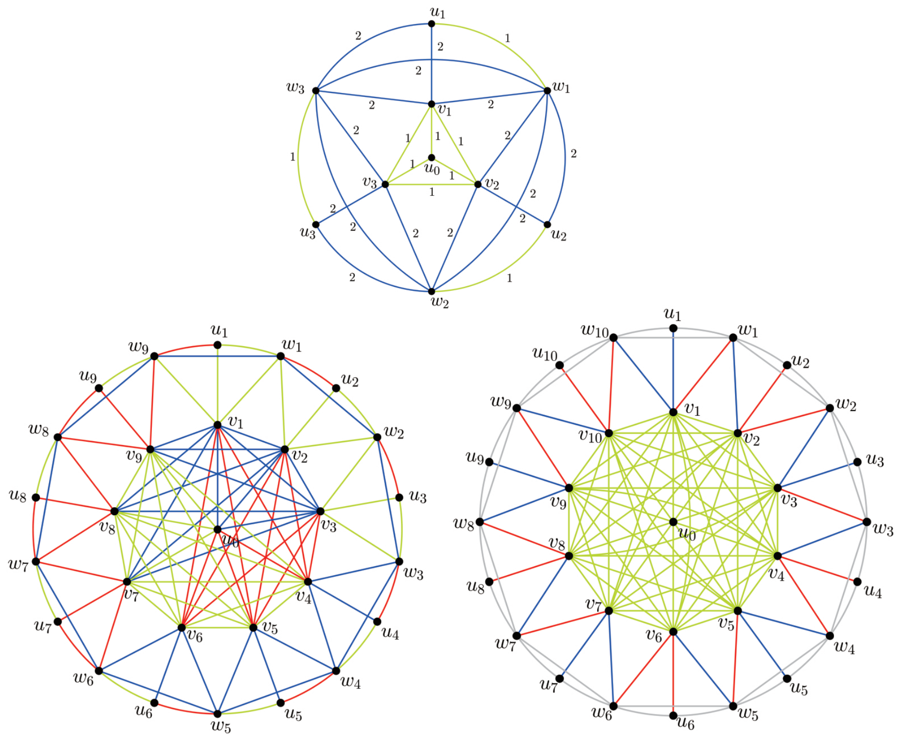

Figure 2. Rainbow coloring of middle wheel W_n, for $n = 3, 9, 10$.

We defined an edge coloring $c_{25} : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$ for middle of wheel W_n for $n \geq 4$ as follows.

(1) If $n \mod 3 \neq 1$,

$$c_{25}(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{w_{i-1}v_i \mid 1 \leq i \leq n, w_0 = w_n\} \cup \{u_i v_i \mid 1 \leq i \leq n\} \cup \\ & \{v_i w_i \mid 1 \leq i \leq n\} \\ f(i), & e \in \{w_{i-1}u_i \mid i \leq i \leq n, w_0 = w_n\} \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \\ & \{w_{i-1}w_i \mid 1 \leq i \leq n, w_0 = w_n\} \\ k, & e \in \{v_i v_j \mid 1 \leq i \leq n, 1 \leq j \leq n, i < j\} \cup \{u_0 v_j \mid 1 \leq j \leq n\} \end{cases}$$

(2) If $n \mod 3 = 1$,

$$c_{25}(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{w_{i-1}v_i \mid 1 \leq i \leq n, w_0 = w_n\} \cup \{u_iv_i \mid 1 \leq i \leq n\} \cup \\ & \{v_iw_i \mid 1 \leq i \leq n\} \\ f(i), & e \in \{w_{i-1}u_i \mid i \leq i \leq n-1, w_0 = w_n\} \cup \{u_iw_i \mid 1 \leq i \leq n-1\} \cup \\ & \{w_{i-1}w_i \mid 1 \leq i \leq n-1, w_0 = w_n\} \\ k, & e \in \{v_iv_j \mid 1 \leq i \leq n, 1 \leq j \leq n, i < j\} \cup \{u_0v_j \mid 1 \leq j \leq n\} \\ 2, & e \in \{w_{n-1}u_n, u_nw_n, w_{n-1}w_n\} \end{cases}$$

Where,

$$f(i) = \begin{cases} i, & \text{if } i = 1,2 \\ i \bmod 3, & \text{if } i \bmod 3 \neq 0 \\ 3, & \text{if } i \bmod 3 = 0 \end{cases}$$

and k is a number which is assigned to $e = v_iv_j$ where $k \neq c(v_iu_i) \neq c(v_ju_j)$.

Theorem 3.3. *If $n \geq 3$ and $G = M(W_n)$ is middle of wheel, then*

$$src(G) = \begin{cases} 2, & n = 3 \\ 3, & 4 \leq n \leq 9 \\ \lceil n/3 \rceil, & n \geq 10 \end{cases}$$

Proof.

(1) Suppose that $n = 3$. Since $rc(G) = 2$ for $n = 3$ in theorem 3.2, then $src(G) \geq 2$. Next, it will show that $src(G) \leq 2$. Since c_{21} is a strong rainbow 2-coloring which is defined in 3.7, it follows that $src(G) = 2$ for $n = 3$.

(2) Suppose that $4 \leq n \leq 9$. Since $rc(G) = 3$ for $4 \leq n \leq 9$, then $src(G) \geq 3$. Next it will shown that $src(G) \leq 3$. Next, to show that $src(G) \leq 3$, we provide a strong rainbow 3-coloring which is defined by c_{25}. Therefore, $src(G) = 3$ for $4 \leq n \leq 9$.

(3) Suppose $n \geq 10$. Then there is an integer z such that $3z - 2 \leq n \leq 3z$. Let G consists of an n-cycle $C_n : u_1, u_2, \ldots, u_n, u_1$ and $V^* = \{v_i \mid 1 \leq i \leq n\}$. First, it will shown that $src(G) \geq z$. Assume, to the contrary, that $src(G) \leq z - 1$. Let c be a strong rainbow $(z-1)$-coloring of G. Since $d(v) = n + 2 > 3(z-1)$ for $v \in V^*$ in G, there exists $V' \subseteq V(C_n)$ such that $|V'| = 4$ and all edges $\{uv \mid u \in V', v \in V^*, \ u \text{ and } v \text{ are adjacent}\}$ are assigned the same. Thus there exist at least two vertices $x, y \in V'$ such that $d(x,y) \geq 3$ in C_n and $d(x,y) = 3$ in G. Let $E' = \{u_iv_i \mid 1 \leq i \leq n\} \subseteq E(G)$. Since x, x', y', y $(xx', yy' \in E')$ is the only $x - y$ geodesic in G, it follows that there is no rainbow $x - y$ geodesic in G, which is a contradiction. Thus $src(G) \geq z$.

Next, to show that $src(G) \leq z$, we provide a strong rainbow z-coloring which is defined by c_{25}. Therefore, $src(G) = \lceil n/3 \rceil$ for $n \geq 10$. □

3.3. *Rainbow Connection of Total Wheel*

Theorem 3.4. *If $n \geq 3$ and $G = T(W_n)$ is total of wheel, then*

$$rc(G) = \begin{cases} 2, & n = 3, 4 \\ 3, & n \geq 5 \end{cases}$$

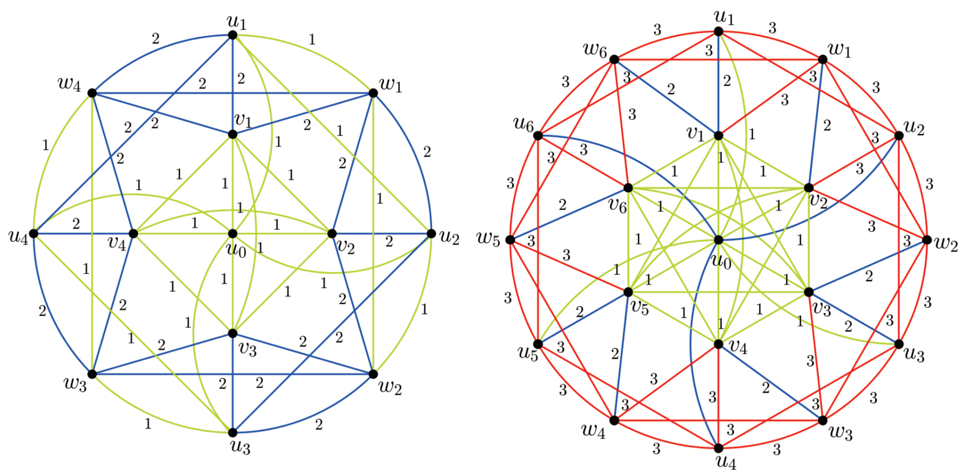

Figure 3. The rainbow coloring of total W_4 and total W_6

Proof.

(1) Suppose that $n = 3, 4$. Since $diam(G) = 2$ for $n = 3, 4$, then $rc(G) \geq 2$. Next, it will shown that $rc(G) \leq 2$. Since $c_{31} : E(G) \to \{1, 2\}$ defined by

$$c_{31}(e) = \begin{cases} 1, & e \in \{u_0u_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq n, i \text{ is odd}, u_{i+1} = u_1\} \cup \\ & \quad \{u_0v_i \mid 1 \leq i \leq n\} \cup \{u_iw_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ & \quad \cup \{w_iw_{i+1} \mid 1 \leq i \leq n, i \text{ is odd}, w_{n+1} = w_1\} \\ 2, & e \text{ others} \end{cases} \tag{3.8}$$

is a rainbow 2-coloring, it follows that $rc(G) = 2$ for $n = 3, 4$.

(2) Suppose that $n \geq 5$. Since $diam(G) = 3$, then $rc(G) \geq 3$. Next, it will shown that $rc(G) \leq 3$. Let $c_{32} : E(G) \to \{1, 2, 3\}$ is a rainbow 3-coloring which is defined by

$$c_{32}(e) = \begin{cases} 1, & e \in \{u_0u_i \mid 1 \leq i \leq n, i \text{ is odd}\} \cup \{u_0v_i \mid 1 \leq i \leq n\} \cup \\ & \quad \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ 2, & e \in \{u_0u_i \mid 2 \leq i \leq n, i \text{ is even}\} \cup \{v_iu_i \mid 1 \leq i \leq n, i \text{ is odd}\} \cup \\ & \quad \{v_iw_{i-1} \mid 1 \leq i \leq n, w_0 = w_n\} \\ 3, & e \text{ others.} \end{cases}$$

For $a, b \in V(G)$, there exists a rainbow 3-coloring c_{32} such as the rainbow $a-b$ path are considered as follow.

(a) If $a = u_i$ dan $b = u_j$

i. $a = u_i, v_0, v_{j-1}, u_j = b$ if i, j are both odd or i, j are both even.

ii. $a = u_i, u_0, u_j = b$ if i is odd and j is even, or i is even and j is odd.

(b) v_i, v_j, u_j if $a = v_i$ and $b = u_j$.

(c) u_i, v_i, v_j, w_j or u_i, v_i, v_{j+1}, w_j if $a = u_i$ and $b = w_j$.

(d) v_i, v_j, w_j if $a = v_i$ and $b = w_j$.

(e) $a = w_i, v_{i+1}, v_j, w_j = b$ if $a = w_i$ and $b = w_j$.

Therefore, $rc(G) = 3$ for $rc(G) \geq 5$ □

We defined an edge coloring $c_{33} : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$ for middle of wheel W_n for $n \geq 5$ as follows.

- If n is even

$$c_{33}(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{u_0u_i \mid 1 \leq i \leq n\} \\ 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{u_iu_{i+1} \mid \\ & \quad 1 \leq i \leq n-1, i \text{ is odd}\} \cup \{w_iw_{i+1} \mid 1 \leq i \leq n-1, i \text{ is odd}\} \\ & \quad \cup\{u_iw_i \mid 1 \leq i \leq n-1, i \text{ is odd}\} \cup \{w_iu_{i+1} \mid 1 \leq i \leq n-1, \\ & \quad i \text{ is odd}\} \\ 2, & e\{u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, i \neq j, w_o = w_1\} \\ & \quad \cup\{u_iu_{i+1} \mid 2 \leq i \leq n, i \text{ is even}, u_{n+1} = u_1\} \cup \{w_iw_{i+1} \mid \\ & \quad 2 \leq i \leq n, i \text{ is even}, w_{n+1} = w_1\} \cup \{u_iw_i \mid 2 \leq i \leq n, \\ & \quad i \text{ is even}\} \cup \{w_iu_{i+1} \mid 2 \leq i \leq n, i \text{ is even}, u_{n+1} = u_1\} \\ 3, & e \text{ others.} \end{cases}$$

- If n is odd

$$c(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{u_0u_i \mid 1 \leq i \leq n\} \\ 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{u_iu_{i+1} \mid \\ & \quad 1 \leq i \leq n-2, i \text{ is odd}\} \cup \{w_iw_{i+1} \mid 1 \leq i \leq n-2, i \text{ is odd}\} \\ & \quad \cup\{u_iw_i \mid 1 \leq i \leq n-2, i \text{ is odd}\} \cup \{w_iu_{i+1} \mid 1 \leq i \leq n-2, \\ & \quad i \text{ is odd}\} \\ 2, & e \in \{u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, i \neq j, w_o = w_1\} \cup \\ & \quad \{u_iu_{i+1} \mid 2 \leq i \leq n-1, i \text{ is even}, u_{n+1} = u_1\} \cup \{w_iw_{i+1} \mid \\ & \quad 2 \leq i \leq n-1, i \text{ is even}, w_{n+1} = w_1\} \cup \{u_iw_i \mid 2 \leq i \leq n-1, \\ & \quad i \text{ is even}\} \cup \{w_iu_{i+1} \mid 2 \leq i \leq n-1, i \text{ is even}, u_{n+1} = u_1\} \\ 3, & e \text{ others.} \end{cases}$$

Theorem 3.5. *If $n \geq 3$ and $G = T(W_n)$ is total of wheel, then*

$$src(G) = \begin{cases} 2, & n = 3, 4 \\ 3, & 5 \leq n \leq 9 \\ \lceil n/3 \rceil, & n \geq 10 \end{cases}$$

Proof.

(1) Suppose that $n = 3, 4$. Since $rc(G) = 2$ for $n = 3, 4$ in theorem 3.4, then $src(G) \geq 2$. Next, it will show that $src(G) \leq 2$. Since c_{31} is a strong rainbow 2-coloring which is defined in 3.8, it follows that $src(G) = 2$ for $n = 3, 4$.

(2) Suppose that $5 \leq n \leq 9$. Since $rc(G) = 3$ for $n \geq 5$ in theorem 3.4, then $src(G) \geq 3$. To show that $src(G) \leq 3$, we provide a strong rainbow 3-coloring which is defined by c_{33}. Therefore, $src(G) = 3$ for $5 \leq n \leq 7$.

(3) Suppose that $n \geq 10$. Then there is an integer k such that $3k - 2 \leq n \leq 3k$. Let G consists of an n-cycle $C_n : u_1, u_2, \ldots, u_n, u_1$. First, it will shown that $src(G) \geq k$. Assume, to the contrary, that $src(G) \leq k - 1$. Let c_{34} be a strong rainbow $(k - 1)$-coloring of G. Since $d(u_0) = 2n > 3(k - 1)$, there exists $V^* \subseteq V(C_n)$ such that $|V^*| = 4$ and all edges $\{uu_0 \mid u \in V^*\}$ are assigned the same. Thus there exist at least two vertices $a, b \in V^*$ such that $d(a, b) \geq 3$ in C_n and $d(a, b) = 2$ in G. Since a, u_0, b is the only $a - b$ geodesic in G, it follows that there is no rainbow $a - b$ geodesic in G, which is a contradiction. Thus $src(G) \geq k$.

Next, to show that $src(G) \leq k$, we provide a strong rainbow k-coloring which is defined by c_{33}. Therefore, $src(G) = \lceil n/3 \rceil$ for $n \geq 10$. □

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